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FLOW OF A VISCOUS LIQUID IN A LAYER ON A ROTATING PLANE

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In carrying out certain modern industrial processes, the application of thin films of uniform thickness onto a flat surface is required. One of the methods used to accomplish this consists in first pouring a sufficiently thick layer of the liquid onto the surface, which then thins out as the specimen is rotated [1]. Similar methods are used in making mirrors [2], color television screens [3], integral schemes, and magnetic memory disks [1]. Rotating disks are also used for spraying and for mixing liquids to accelerate heterogeneous chemical reactions in various processes of chemical technology [4-6].

To effectively control these processes one needs to know the nature of the flows that arise. Since the radius of the rotating disk is usually many times the thickness of the liquid layer, one can, for the purposes of mathemetical modelling, replace the disk by an infinite rotating plane. In the present paper we construct stationary and self-similar Karman-type solutions of the Navier—Stokes equations, which describe the flow of a viscous liquid in the layer between a rotating solid plane and the free surface parallel to it.

1. Statement of the Problem. We consider a rotationally-symmetric flow of a viscous incompressible liquid in a layer $\Lambda_t = \{(r, \theta, z) \in R^3,\ z \in (0, Z(t))\}$, bounded above by its free surface and below by a solid wall rotating around the z-axis at a given angular rate $\Omega(t)$.

The field of velocity and pressure in the liquid [$V(r, z, t)$ and $p(r, z, t)$] satisfies the Navier—Stokes equations

$$\begin{gathered}u_t + uu_r - r^{-1}v^2 + wu_z = -\rho^{-1}p_r + \nu[u_{rr} + (r^{-1}u)_r + u_{zz}],\\ v_t + uv_r + r^{-1}uv + wv_z = \nu[v_{rr} + (r^{-1}v)_r + v_{zz}],\\ w_t + uw_r + ww_z = -\rho^{-1}p_z + \nu[w_{rr} + r^{-1}w_r + w_{zz}],\\ u_r + r^{-1}u + w_z = 0\end{gathered} \tag{1.1}$$

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in the region Λ_t and the adherence boundary conditions

$$u = 0,\ v = r\Omega(t),\ w = 0 \quad \text{for} \quad z = 0, \tag{1.2}$$

and also the dynamic

$$p = p_0 + 2\rho\nu w_z,\ u_z = v_z = 0 \tag{1.3}$$

and kinematic

$$w = dZ/dt \tag{1.4}$$

conditions on the free boundary $z = Z(t)$. Here u, v, w are, respectively, the radial, circumferential, and axial components of velocity $\mathbf{V}$; ρ is the density of the liquid; ν is the kinematic coefficient of viscosity; a subscript indicates partial differentiation with respect to the corresponding argument.

Further, we consider solutions of Eqs. (1.1)-(1.4) for which functions u and v are linear in the variable r, and for which p and w are independent of r. Let

$$u = r\Omega_0 F(\xi, \tau),\ v = r\Omega_0 G(\xi, \tau),\ \Omega = \Omega_0\omega(\tau),$$
$$w = \sqrt{\nu\Omega_0}H(\xi, \tau),\ p = \rho\nu\Omega_0 Q(\xi, \tau), \tag{1.5}$$

where $\xi = z\sqrt{\Omega_0/\nu}$; Ω_0 is a characteristic angular rate; $\tau = \Omega_0 t$. Then Eqs. (1.1) assume the form

$$F_\tau + HF_\xi + F^2 - G^2 = F_{\xi\xi}; \tag{1.6}$$
$$G_\tau + HG_\xi + 2FG = G_{\xi\xi}; \tag{1.7}$$
$$2F + H_\xi = 0; \tag{1.8}$$
$$H_\tau + HH_\xi = -Q_\xi + H_{\xi\xi} \tag{1.9}$$

for $\xi \in (0, D(\tau))$, $\tau > 0$ $[D(\tau) = \sqrt{\Omega_0/\nu}Z]$, and the boundary conditions (1.2)-(1.4) become:

$$Q = 2H_\xi + p_0/(\rho\nu\Omega_0); \tag{1.10}$$
$$F = 0,\ G = \omega(\tau),\ H = 0 \quad \text{for} \quad \xi = 0; \tag{1.11}$$
$$F_\xi = G_\xi = 0; \tag{1.12}$$
$$H = dD/d\tau \quad \text{for} \quad \xi = D(\tau). \tag{1.13}$$

The problem is completed with the assignment of the initial data

$$F = F^0(\xi),\ G = G^0(\xi),\ H = H^0(\xi) \quad \text{for} \quad \xi \in (0, D_0),$$
$$\tau = 0,\ D(0) = D_0. \tag{1.14}$$

It is readily seen that problem (1.6)-(1.14) breaks down into two problems, solvable in succession. Functions $F(\xi, \tau)$, $G(\xi, \tau)$, $H(\xi, \tau)$, $D(\tau)$ are determined by solving the closed problem (1.6)-(1.8), (1.11)-(1.14); the pressure $Q(\xi, \tau)$ is then recovered after this by solving equations (1.9) and (1.10).

Solutions of the form (1.5) were first considered by Karman [7]. The initial-boundary problem (1.6)-(1.14) was investigated in [1, 8]. Its unique solvability in the small with respect to the time was proved in [8] for smooth functions $\omega(\tau)$ upon satisfaction of conditions of compatibility of the initial data and boundary conditions. In [1] a problem was considered with the incompatible initial and boundary conditions $\omega = \omega_0$, $F^0(\xi) = G^0(\xi) = H^0(\xi) = 0$ and a formal asymptotic expansion was formulated for small Reynolds numbers ($\mathrm{Re} = D_0^2\omega_0 = \Omega(0)Z^2(0)/\nu$) and small values of τ. The stationary problem (1.6)-(1.12) was solved numerically in [4-6] for the given value $D(\tau) = D_0$. The kinematic condition (1.13) was not satisfied for all solutions constructed therein, so that its physical interpretation was clouded.

The present paper is devoted to the construction of stationary and self-similar solutions of the problem (1.6)-(1.13). Let $\omega_n(\tau) = (1 + n\tau)^{-1}\omega_n$ for $n = -1, 0, 1$; the problem (1.6)-(1.13) then admits solutions of the form

$$H(\xi, \tau) = (1 + n\tau)^{-1/2}[h_n(\zeta_n) + n\zeta_n/2],$$

$$F(\xi, \tau) = (1 + n\tau)^{-1} f_n(\zeta_n),\ G(\xi, \tau) = (1 + n\tau)^{-1} g_n(\zeta_n),$$
$$D_n(\tau) = (1 + n\tau)^{1/2} d_n,\ \zeta_n = (1 + n\tau)^{-1/2}\xi.$$

The unknown functions $f_n(\zeta)$, $g_n(\zeta)$, $h_n(\zeta)$ and the number d_n satisfy for various n the equations (subscripts n are omitted; a prime denotes differentiation with respect to the variable ζ):

$$f'' = f'h + f^2 - g^2 - nf; \tag{1.15}$$
$$g'' = g'h + 2fg - ng; \tag{1.16}$$
$$h' = -2f - n/2 \tag{1.17}$$

and boundary conditions

$$f(0) = h(0) = 0,\ g(0) = \omega; \tag{1.18}$$
$$f'(d) = g'(d) = h(d) = 0. \tag{1.19}$$

When n = −1 these solutions describe a self-similar spreading-out flow; when n = 0 they describe a stationary solution; and for n = 1 they describe a self-similar regime for a thickened layer on the rotating plane.

2. Flow in a Layer on a Fixed Plane. If plane z = 0 is not rotating, i.e., if $\omega = 0$, system (1.14)-(1.16) then has solutions such that $g \equiv 0$, and functions f and h satisfy the equations

$$f'' = f'h + f^2 - nf; \tag{2.1}$$
$$h' = -2f - n/2. \tag{2.2}$$

Boundary conditions in this case assume the form

$$f(0) = h(0) = f'(d) = h(d) = 0. \tag{2.3}$$

Let $f(\zeta)$, $h(\zeta)$, d be a solution of problem (2.1)-(2.3). Integrating equation (2.2) from 0 to d and taking account of the boundary conditions, we obtain

$$\int_0^d f(\zeta)\, d\zeta = -nd/4. \tag{2.4}$$

It follows from equations (2.1) and (2.2) that

$$(f + h^2/4)'' = n^2/8 + 3f^2 \geqslant n^2/8. \tag{2.5}$$

Integrating this latter inequality from d to ξ, and then from 0 to ζ, and taking account of boundary conditions (2.3), we obtain

$$f + h^2/4 \leqslant n^2(\zeta^2 - 2d\zeta)/16, \tag{2.6}$$

which, after substitution into Eq. (2.4), yields

$$-6nd \leqslant -n^2d^3. \tag{2.7}$$

Equality in relations (2.5)-(2.7) is possible only if $f \equiv 0$.

The following propositions hold:

1. If $\omega = 0$, $g \equiv 0$, n = 0, then $f \equiv 0$, $h \equiv 0$, i.e., there are no nontrivial Karman-type flows on a fixed plane. Otherwise we would have inequality (2.7) as a strict inequality, i.e., 0 < 0.

2. Problem (2.1)-(2.3) has no solutions when n = −1. Otherwise inequality (2.7) would be satisfied, i.e., $6d \le -d^3$.

3. If n = 1, the solution of problem (2.1)-(2.3) satisfies the inequalities

$$d < \sqrt{6}; \tag{2.8}$$

$$-2.3 < f'(0) < 0. \tag{2.9}$$

Inequality (2.8) follows from inequality (2.7) with n = 1. To prove relation (2.9), it is necessary to integrate relation (2.5) from 0 to d and then use relations (2.3). This gives

$$f'(0) < 0. \tag{2.10}$$

This establishes the right side of inequalities (2.9).

It follows from the estimate (2.6) and equation (2.1) that f cannot attain a maximum for $\zeta \in [0, d]$. Therefore, by virtue of relations (2.3) and (2.10), $f(\zeta)$ decreases monotonically, and since $h'' = -2f'$, function $h(\zeta)$ is convex downwards and, since $h(0) = h(d) = 0$, it is negative for $\zeta \in (0, d)$.

Taking into account the properties of the functions $f(\zeta)$, $h(\zeta)$ established above and noting the form of Eq. (2.1), it is easy to show that $f''(\zeta) > 0$ for $\zeta \in (0, d)$, whence $f(\zeta) < f(d)\zeta/d$. Substituting the latter inequality into Eq. (2.4), we have $f(d) > -1/2$. Since $f'(\zeta) < 0$, the latter inequality implies that

$$f(\zeta) > f(d) > -1/2. \tag{2.11}$$

Since $f(\zeta) < 0$, it follows from inequalities (2.11) that

$$f^2(\zeta) < 1/4 \quad \text{for} \quad \zeta \in (0, d). \tag{2.12}$$

Substitution of inequality (2.12) into relation (2.5) yields $(f + h^2/4)'' = 3f^2 + 1/8 < 7/8$. Integrating this latter inequality from 0 to d and using relations (2.3), we can show that $f'(0) > -7d/8$, whence, by virtue of the fact that d satisfies inequality (2.8), we see that inequality (2.9) is valid.

3. Self-Similar Regime of Thickened Layer on a Rotating Plane. Numerical Solution. Let $\mathbf{F}_\omega(x, y, d) = (f'(d), g'(d), h(d))$, where $f(\zeta)$, $g(\zeta)$, $h(\zeta)$ is a solution of the Cauchy problem

$$f(0) = h(0) = 0,\ g(0) = \omega,\ f'(0) = x,\ g'(0) = y \tag{3.1}$$

for the system of equations (1.15)-(1.17) with n = 1. Then the solution of the boundary value problem (1.15)-(1.19) is equivalent to the solution of the system of three equations

$$\mathbf{F}_\omega(x, y, d) = 0 \tag{3.2}$$

with three unknowns $(x, y, d) = \mathbf{X}$.

It was shown above that when $\omega = 0$ the solution of system (3.2) with $y = 0$ must be sought in the region $x \in (-2.3; 0)$, $d \in (0; 2.5)$. This solution was obtained as part of her course work by student S. B. Barabanova of Novosibrisk State University (using a method of adjustment with respect to parameters x and d). It turned out that $x = x^0 \simeq -1.4006$, $d = d^0 \simeq 1.3231$.

Graphs of functions $f^0(\zeta)$ and $a^0(\zeta) = h^0(\zeta) + \zeta/2$ are shown in Figs. 1 and 2 (curves 1). Function $a(\zeta)$ is proportional to the axial flow rate; $f(\zeta)$ is proportional to the radial flow rate. The solution constructed describes the flow in a thickening layer on a fixed plane stipulated by liquid inflow from infinity. When $\omega > 0$, solutions of Eq. (3.2) are sought using the following method of continuation with respect to the parameter ω. Assume that a solution $\mathbf{X}_{j-1}$ of Eq. (3.2) is known for $\omega = \omega_{j-1}$; then the solution for $\omega = \omega_j = \omega_{j-1} + \Delta\omega_j$ may be found with the aid of an iterational process (modified Newton method):

$$\mathbf{X}_j^0 = \mathbf{X}_{j-1} + \Delta\omega_j (\partial\mathbf{F}_\omega/\partial\mathbf{X})^{-1}\,\partial\mathbf{F}_\omega/\partial\omega; \tag{3.3}$$

$$\mathbf{X}_j^{k+1} = \mathbf{X}_j^k - (\partial\mathbf{F}_\omega/\partial\mathbf{X})^{-1}\cdot\mathbf{F}_\omega(\mathbf{X}_j^k); \tag{3.4}$$

$$\mathbf{X}_j = \lim_{k\to\infty}\mathbf{X}_j^k, \tag{3.5}$$

where $(\partial\mathbf{F}_\omega/\partial\mathbf{X})^{-1}$ are matrices inverse to $(\partial\mathbf{F}_\omega/\partial\mathbf{X})$, calculated for $\mathbf{X} = \mathbf{X}_{j-1}$, $\omega = \omega_{j-1}$ and for $\mathbf{X} = \mathbf{X}_j^0$, $\omega = \omega_j$ in relations (3.3) and (3.4), respectively.

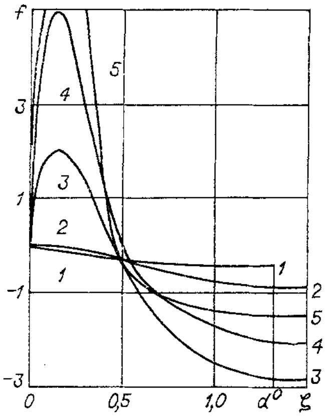

Fig. 1

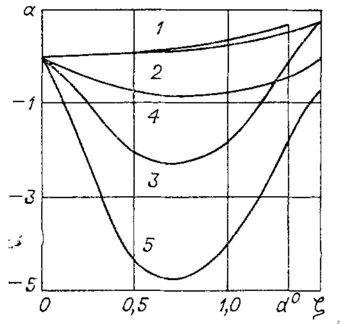

Fig. 2

In implementing this method numerically we obtained the vector-valued function $\mathbf{F}_\omega(\mathbf{X})$ by replacing equations (1.15)-(1.17) by a difference scheme of second order accuracy with step $\Delta\zeta = 0.001$. Partial derivatives of $\mathbf{F}_\omega$ (apart from $\partial\mathbf{F}_\omega/\partial d$) in Eqs. (3.3) and (3.4) were replaced by difference analogs of the form

$$\frac{\partial\mathbf{F}_\omega}{\partial x} = \frac{\mathbf{F}_\omega(x+\Delta x) - \mathbf{F}_\omega(x-\Delta x)}{2\Delta x}$$

and similarly ($\Delta x = \Delta y = \Delta\omega = 10^{-4}$). We put $\partial\mathbf{F}_\omega/\partial d = (f''(d),\ g''(d),\ h'(d))$. Condition (3.5) was replaced by $\mathbf{X}_j = \mathbf{X}_j{}^k$ if $|\mathbf{X}_j^k - \mathbf{X}_j^{k+1}| < 10^{-8}$. Values of $\Delta\omega_j$ were chosen as follows:

$$\Delta\omega_j = \begin{cases} 0.01 & \text{for} \quad \omega_j \in [0, 2) \cup [3, 5), \\ 0{,}001 & \text{for} \quad \omega_j \in [2, 3), \\ 0.1 & \text{for} \quad \omega_j \in [5, 10), \\ 1 & \text{for} \quad \omega_j \in [10, 100). \end{cases}$$

Results of the calculations are shown in Figs. 1-4. In Fig. 4 the continuous curve is the graph of the function $d(\omega)$. It turns out that $d(\omega)$ increases for $\omega < \omega^0 \simeq 1.53$ and decreases for $\omega > \omega^0$, a result which is connected with a change in the nature of the flow. For $\omega < \omega_0 \simeq 1.06$ the axial component of the velocity is everywhere positive, the radial component is everywhere negative, and the circumferential velocity varies in sign. Close to the free surface the liquid rotates in a direction counter to the direction of rotation of the solid surface. The absolute value ω^1 of the angular rate of the liquid on the free surface is close to 2ω for small values of ω. As ω grows, the ratio ω^1/ω decreases, ω^1 increases. Shown in Figs. 1-3 are the graphs of $f(\zeta)$, $a(\zeta)$, and $g(\zeta)$, respectively, for $\omega = 0.2$ (curves 2).

The at-first-glance paradoxical nature of these solutions may be explained by the fact that the reason for the motion of the liquid here, along with the rotation of the disk, is the flow at infinity. For small ω centrifugal forces on the rotating disk are insufficient for the flow to spread.

When $\omega > \omega_0$ a zone appears close to the solid surface where $a(\zeta) < 0$, $f(\zeta) < 0$, i.e., the liquid is driven back by centrifugal forces. In a neighborhood of the free boundary the liquid, as in the case of small ω, moves towards the center and upwards and rotates in a direction counter to the direction of rotation of the disk; ω^1 increases with an increase in ω, ω^1/ω decreases. In Figs. 1-3 curves 3 are the graphs of functions $f(\zeta)$, $a(\zeta)$, and $g(\zeta)$ for $\omega = 14.5$. "Counter-rotation" of the liquid close to the free surface is, apparently, a consequence of the self-similarity of the solutions considered, requiring for its realization special conditions at infinity.

Solutions of problem (1.15)-(1.19) were constructed by the method described above for $\omega \le 100$. If $\omega > 100$, this method does not insure the accuracy needed since the values of the derivatives of the unknown functions become too large. For large values of ω it is convenient to make calculations in new variables, which we introduce below.

4. Asymptotics of Solutions for Large Angular Rates. Stationary Solutions. The change of variables

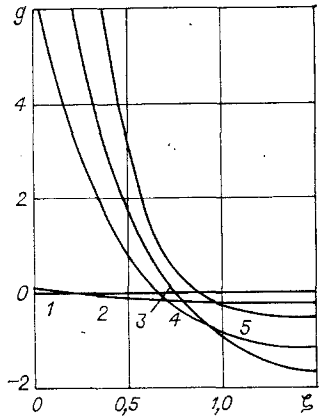

Fig. 3

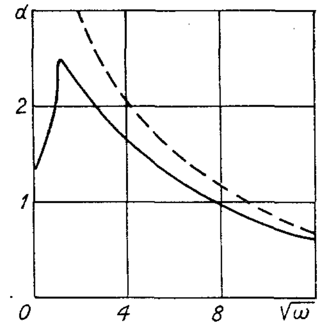

Fig. 4

$$\zeta = d\eta,\ f = \varphi d^{-2},\ g = \psi d^{-2},\ h = \chi d^{-1}; \tag{4.1}$$

$$\alpha = d^2 \tag{4.2}$$

brings problem (1.15)-(1.19) to the form

$$\varphi'' = \chi\varphi' + \varphi^2 - \psi^2 - \alpha\varphi,\ \psi'' = \chi\psi' + 2\varphi\psi - \alpha\psi,\quad \chi' = -2\varphi - \alpha/2; \tag{4.3}$$

$$\varphi(0) = \chi(0) = 0,\ \psi(0) = \psi_0 = \omega d^2,\ \varphi'(1) = \psi'(1) = \chi(1) = 0. \tag{4.4}$$

Parameter ψ_0 below is considered as given; α is an unknown.

It is obvious that an arbitrary solution of problem (4.3) and (4.4) such that $\alpha > 0$ after the change of variables (4.1) and (4.2) yields the solution of problem (1.15)-(1.19) with n = 1, and, conversely, a change of variables, inverse to relations (4.1) and (4.2), takes an arbitrary solution of problem (1.15)-(1.19) into a solution of problem (4.3) and (4.4) with $\alpha > 0$.

Problem (4.3) and (4.4) was solved numerically by a method of continuation with respect to parameter ψ_0 similar to the method described in Sec. 3. It was assumed that $\mathbf{F}_{\psi_0}(\mathbf{X}) = (\varphi'(1), \psi'(1), \chi(1))$, where $\mathbf{X} = (\varphi_0', \psi_0', \alpha)$, $\varphi(\eta)$, $\psi(\eta)$, $\chi(\eta)$ is a solution of the Cauchy problem $\varphi'(0) = \varphi'_0, \psi'(0) = \psi'_0$, $\psi(0) = \psi_0$ for equation (4.3), $\mathbf{X}_0 = (x(100)d^3(100),\ y(100)d^3(100),\ d^2(100))$.

The calculations made showed that $\alpha > 0$ for $\psi_0 < \psi_{0*} \simeq 53.73$, $\alpha(\psi_{0*}) = 0$. Functions φ_*, ψ_*, χ_* are a solution of equations (4.3) and (4.4) for $\psi_0 = \psi_{0*}$. Returning to the variables f, g, h, d, ζ, we can conclude that $d^2\omega \to \psi_{0*}$, $f(\zeta)/\omega \to \varphi_*(\sqrt{\psi_{0*}/\omega}\,\zeta)/\psi_{0*}$, $g(\zeta)/\omega \to \psi_*(\sqrt{\psi_{0*}/\omega}\,\zeta)/\psi_{0*}$, $h(\zeta)/\sqrt{\omega} \to \chi_*(\sqrt{\psi_{0*}/\omega}\,\zeta)/\sqrt{\psi_{0*}}$ as $\alpha \to \infty$.

It is not hard to see that if $\alpha = 0$ the system (4.3) coincides with the system (1.15)-(1.17) for n = 0. Therefore the solution $\varphi_*(\zeta)$, $\psi_*(\zeta)$, $\chi_*(\zeta)$, 0 for problem (4.3) and (4.4) that we have constructed is at the same time a solution of problem (1.15)-(1.19) with n = 0, d = 1, i.e., a stationary solution of problem (1.6)-(1.13).

It is easily verified that when n = 0 the change of variables (4.1) and (4.2) transforms Eqs. (1.15)-(1.17) and the homogeneous boundary conditions into themselves. The right side of the nonhomogeneous boundary condition transforms into ωd^2. Therefore, for an arbitrary value of ω and n = 0 problem (1.15)-(1.19) has the solution

$$f_\omega(\zeta) = \omega\varphi_*(d_0(\omega)\zeta)/\psi_{0*},\quad h_\omega(\zeta) = \omega^{1/2}\chi_*(d_0(\omega)\zeta)/\psi_{0*}^{1/2},$$

$$g_\omega(\zeta) = \omega\psi_*(d_0(\omega)\zeta)/\psi_{0*}\quad (d_0(\omega) = (\psi_{0*}/\omega)^{1/2}).$$

Graphs of the functions $f_\omega(\zeta)$, $a_\omega(\zeta)$, $g_\omega(\zeta)$ for $\omega = 23.88$ appear as curves 4 in Figs. 1-3, respectively. The graph of function $d_0(\omega)$ is shown in Figs. 4 and 5 by the dashed curve.

Remark 4.1. If we extend functions $f(\zeta)$ and $g(\zeta)$ symmetrically with respect to the line $\zeta = d$ and extend $h(\zeta)$ anti-symmetrically, we obtain functions satisfying for $\zeta \in (0, 2d)$ the equations (1.15)-(1.17), and satisfying for $\zeta = 0$ and $\zeta = 2d$ the boundary conditions (1.18) on the solid wall, i.e., rotating in one direction and with identical angular rates.

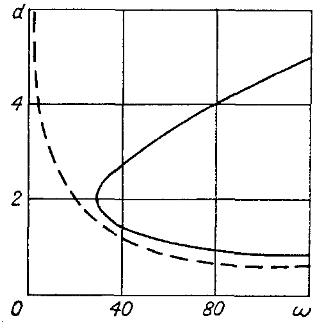

Fig. 5

Remark 4.2. For all the stationary solutions constructed we have $Re = \Omega Z^2/\nu = \psi_{0*} \simeq 53.73$.

5. Spreading of a Layer on the Rotating Plane. Problem (4.3) and (4.4) turns out to be solvable even when $\psi_0 > \psi_{0*}$; however, $\alpha < 0$ when $\psi > \psi_{0*}$. Let $\varphi(\eta)$, $\psi(\eta)$, $\chi(\eta)$, $\alpha < 0$ be a solution of problem (4.3) and (4.4). If we put $d^2 = -\alpha$ and make the change of variables (4.1), the resulting functions $f(\zeta)$, $g(\zeta)$, $h(\zeta)$ and number d will be a solution of problem (1.15)-(1.19) with $n = -1$, i.e., they will describe a self-similar regime for spreading of the layer on the rotating plane.

We solved problem (4.3), (4.4) numerically for $\psi_0 > \psi_{0*}$, using the same method as for $\psi_0 < \psi_{0*}$. In these calculations the value of α varied from 0 to -1, ω from ∞ to 50, and d from 0 to 1. For small values of ω the solution was carried out in the variables ζ, f, g, h, d by the method described in Sec. 3. Solutions were obtained for $\omega > \omega_* \simeq 30.68$. It turned out to be the case that $|d'(\omega)| \to \infty$ as $\omega \to \omega_*$, and it was not possible to continue the solution for $\omega < \omega_*$. The last equation furnished the basis for our assumption that the function $d(\omega)$ is not unique. Succeeding calculations confirmed this assumption. These were made by the method described in Sec. 3; however, in problem (1.15)-(1.19) d was considered to be given and ω was considered as an unknown. We used the method of continuation with respect to parameter d. We assumed that $\Delta d = 0.1$, $d^0 = d(40)$, $x^0(40)$, $y^0 = y(40)$, $\omega^0 = 40$. Quantities appearing in the right-hand members were calculated earlier by the method of continuation with respect to the parameter ω.

The graph of the function $d(\omega)$ appears as the dashed curve in Fig. 5. Solutions corresponding to the lower part of this graph describe flows such that close to the solid plane the liquid rotates in the same direction as the disk and spreads out along it. Close to the free surface the liquid flows toward the center and rotates in a direction counter to the direction of rotation of the disk. As ω increases, ω^1 and ω^1/ω increase. Graphs of the functions $f(\zeta)$, $g(\zeta)$ and $a(\zeta)$ for $\omega = 37.94$ appear as curves 5 in Figs. 1-3. For solutions corresponding to the upper part of the graph in Fig. 5, ω^1 and ω^1/ω decrease as ω increases. For large ω these solutions are of the nature of a boundary layer; outside of a narrow zone close to the rotating plane they are close to

$$f(\zeta) = -1,\ g(\zeta) = 0,\ h(\zeta) = 5(\zeta - d)/2. \tag{5.1}$$

Functions $f(\zeta)$, $g(\zeta)$ and $h(\zeta)$, defined by formulas (5.1), satisfy equations (1.15)-(1.17) and the conditions on the free boundary $\zeta = d$ for an arbitrary value of d, which increases almost linearly with increasing ω. Calculations were carried out up to $\omega = 50$.

The solutions constructed describe flows for which the thickness D of the layer varies in accordance with the law $D = d\sqrt{1-\tau}$. After a finite time $D(\tau)$ vanishes, i.e., the surface "dries up". These solutions, apparently, do not exhaust the whole class of solutions of problem (1.15)-(1.19). For example, solutions can exist which have for large values of ω the asymptotics, not of formulas (5.1), but $f(\zeta) = 0$, $g(\zeta) = 0$, $h(\zeta) = (\zeta - d)/2$.

We remark that from the solutions constructed in Secs. 3-5 we can obtain, using a dilatation transformation of the variables F, G, H, ξ, τ, self-similar solutions of problem (1.5)-(1.13) of the general form

$$F = (a + b\tau)^{-1} f(\zeta),\ G = (a + b\tau)^{-1} g(\zeta),\ H = (a + b\tau)^{-1/2} a(\zeta),$$
$$\zeta = \xi/\sqrt{a + b\tau}.$$

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RAREFIED GAS MOTION IN A SHORT PLANAR CHANNEL OVER THE ENTIRE KNUDSEN NUMBER RANGE

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It was demonstrated in [1] that flow of a rarefied gas in a finite channel has been considered only over a narrow Knudsen number range or in coarse approximations valid only for sufficiently long channels. In that study the problem was solved for a wide range, but in the approximation that molecules entering the channel through its faces have an absolutely Maxwellian distribution function, which also limits application of its results to finite, although sufficiently long channels. In connection with this there is a need for a precise solution of the given problem over the entire range of Knudsen numbers with consideration of flow formation in the region of the vessel near the input.

1. We will consider a planar channel of length ℓ, height $2a$, infinite in the z-direction, connecting two semi-infinite vessels of one and the same gas (Fig. 1). Within the vessels at a sufficient distance from the channel the gas is maintained under equilibrium conditions at pressures p_1 and p_2 and identical temperatures T. Under the action of the pressure head the gas moves in the x-direction.

We introduce the scale factors: a, n_1, $\beta^{1/2} = (2RT)^{1/2}$, $n_1\beta^{-3/2}$, $\eta_1 = n_1 m v \lambda_1/2$ for the length, density n, velocities $\mathbf{c}$ and $\mathbf{u}$, distribution function f, and viscosity coefficient η. Here R is the ideal gas constant, m is the mass of a molecule, $v = (8RT/\pi)^{1/2}$ is the thermal velocity of a molecule, λ_1 is the molecular free path length in the first vessel. All further expressions will be written using these scaling factors.

We assume that the relative pressure head is much less than unity ($|p_2 - p_1|/p_1 \ll 1$) and that all gas molecules are reflected from the walls of the channel and vessels diffusely. For the distribution function equation we use the BGK model of the Boltzmann equation [2]

$$\mathbf{c}\partial f/\partial \mathbf{r} = \delta(f^0 - f),$$

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